Fusion or Integration: What's the difference?

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Abstract – *The U. S. Air Force*[1] *uses the term "fusion" in a very specific manner. For example, the U.S. Air Force Research Laboratories have defined fusion on different objects, like sensors, data and classifiers. Yet there is ambiguity in some instances as to what is meant by it usage. Other Air Force research and acquisitions groups use the term "integration" to describe the process of combining data, knowledge, command, control, intelligence, surveillance, and reconnaissance. Even the U.S. Defense Advanced Research Program Agency (DARPA) has a program called "Integrated Sensing and Processing" (ISP) that aims to open the next paradigm for application of mathematics to the design and (co)operation of DoD sensor/exploitation systems and networks of such systems. The program hopes to develop mathematical tools that enable the design and global optimization of systems that interactively combine traditionally independent functions of sensing, signal processing, communication, and exploitation. On the surface it appears that integration is the same as fusion. In this paper, we define fusion and integration using the language of category theory. These definitions are in agreement with their usage in the Air Force. Using category theory we show the difference (and similarities) between fusion and integration.*

Keywords: Fusion, integration, category theory, graph theory.

1 Introduction

Several definitions have been given in the literature to capture the essence of fusion. Over the years these definitions have improved as we learned more about fusion processes. Some definitions of fusion use integration to qualify the definition. Does this mean that integration and fusion are the same? Are fusion and integration synonyms? Integration, on the other hand, does not have several definitions floating around the literature. Yet the word is used profusely. Webster's Dictionary defines "integrate" as

> **integrate** *v.* *–tr.* **1**. To make into a whole by bringing all parts together; unify.
> **integration** *n.* **1**. The act or process of integrating.

The use of the term "integration" in the context of fusion appears to be routine. That is, fusion is defined using integration. Why? The Journal of Information Sciences has a special issue on web data integration [**?**] with a few papers discussing information integration[**?**].

The Defense Advanced Research Program Agency (DARPA) has a program called *Integrated Sensing and Processing* (ISP) where the goal is to "open the next paradigm for application of mathematics to the design and (co)operation of DoD sensor/exploitation systems and networks of such systems"[**?**]. The program hopes to develop mathematical tools that enable the design and global optimization of systems that interactively combine traditionally independent functions of sensing, signal processing, communication, and exploitation.

The U.S. DoD Command and Control (C2) Enterprise Reference Architecture (1 December 2002) v3.0-14 defines:

> integration – identifies functional and data commonalities among systems and eliminates redundancy by aggregating these aspects into a reduced number of modernized data systems in a shared environment. Bringing applications into the shared environment can include some or all of the following; moving systems onto a common infrastructure; sharing data to provide a single logical authoritative source, view and interpretation of data; sharing code for common functions that can reduce code maintenance costs across separate applications; and standardization of user interfaces to provide a common look and feel.

This is a long and involved definition. The U.S. Air Force Warfighter C2ISR Integration Page at https://afc2isrc.af.mil/warfighter has another definition.

So, we ask the question "does fusion and integration mean the same?" If they do then we should NOT define fusion in terms of integration and vice versa. When writing a definition care must be taken so that the new term is defined using well understood and well-defined terms.

1.1 Motivation

Our motivation for answering this question is two fold: (1) Based upon the similarities, researchers wishing to "integrate" can use existing fusion techniques. Researchers may not be aware of the many fusion techniques that would help

[1] The views expressed in this article are those of the authors and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the US Government.

them. (2) Based upon the differences, new fusion techniques may have to be created to handle "integration" problems. But, to do this we need to know "what is integration?"

1.2 Problem Statement

Can the fusion and integration processes be described in a unifying mathematical manner? We believe the branch of mathematics known as category theory [1, 2, 3], provides a way to describe the processes. Once a process description is attained, the relationships that exist in fusion processes may be explored using the algebraic theorems and properties of category theory.

In this paper we will define fusion and integration using well-defined mathematical terminology. We will show the differences and similarities of fusion and integration.

2 Background

In this section we give a short literature review of fusion and integration. Then a subsection covering category theory follows. We do not wish the reader to think that we are being harsh in the following critique of the definitions. We wish to convene the sense that the definitions have improved over the years as the community has learned more and understood more about fusion processes. Even our definition of integration may need to change as we understand more about integration processes.

In mathematics, one writes a definition using well-defined terms to avoid circularity. Although following these terms to their origins will result in terms that are defined using undefined terms, like "set", "element" and "is an element of". These are undefined, but assumed to be understood. Writing a definition one may use these terms (and terms derives from these, like a mapping) and circularity will be avoided.

2.1 Fusion

The literature on fusion is vast in the community of information fusion and sensor fusion. The definition of data fusion as defined by the Department of Defense of the United States of America, Data Fusion Subpanel of the Technology Panel for C3 (command, control, communications) of the Joint Directors of Laboratories (JDL) [**?**]as a "multilevel, multifaceted process dealing with the automatic detection, association, correlation, estimation, and combination of data and information from single and multiple sources ". This definition is more general than the previous one [**?**] with respect to the types of information than can be combined.

Other use of fusion has occurred in computer science community where they specific "horizontal" fusion and "vertical" fusion as well.

2.2 Category Theory

It is apparent that using directed graphs will be very useful in developing a model of fusion. The branch of mathematics known as Category Theory quite naturally takes advantage of this. In fact, the basic definition of a category includes a definition of a directed graph as well. Other useful elements will become apparent later, but exploring the full power of category theory in order to produce a theory of fusion is part of the research. In this section, we have drawn upon various authors' presentations to explain the basics of category theory [4, 5, 2, 3].

Definition 1 (Category) *A category $\mathcal{C}$ consists of the following:*

A1. A collection of objects denoted $\mathbf{Ob}(\mathcal{C})$.

A2. A collection of maps denoted $\mathbf{Ar}(\mathcal{C})$.

A3. Two mappings, called Domain (dom) and Codomain (cod), which assign to an arrow $f \in \mathbf{Ar}(\mathcal{C})$ a domain and codomain from the elements of $\mathbf{Ob}(\mathcal{C})$. Thus, for arrow f, given by $O_1 \xrightarrow{f} O_2$, $dom(f) = O_1$ and $cod(f) = O_2$.

A4. A mapping assigning each object $\mathrm{O} \in \mathbf{Ob}(\mathcal{C})$ an unique arrow 1_{O} called the identity arrow, such that

$$\mathrm{O} \xrightarrow{1_{\mathrm{O}}} \mathrm{O}$$

and such that for any existing element, x, of O, we have that

$$x \xmapsto{1_{\mathrm{O}}} x.$$

A5. A map, $\circ$, called composition, $\mathbf{A} \times \mathbf{A} \xrightarrow{\circ} \mathbf{A}$. Thus, given $f, g \in \mathbf{A}$ with $cod(f) = dom(g)$ there exists an unique $h \in \mathbf{A}$ such that $h = g \circ f$.

Notice that Axioms A1 - A3 define a directed graph, where the objects are the nodes and the arrows are the directed edges of the graph. Axioms A3-A5 lead to the associative and identity rules:

- **Associative Rule**. Given appropriately defined arrows f, g, and h we have that

$$(f \circ g) \circ h = f \circ (g \circ h).$$

- **Identity Rule**. Given arrows $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$, then there exists 1_A such that $1_A \circ g = g$ and $f \circ 1_A = f$.

Definition 2 (Subcategory) *A subcategory $\mathcal{B}$ of $\mathcal{A}$ is a category whose objects are some of the objects of $\mathcal{A}$ and whose arrows are some of the arrows of $\mathcal{A}$, such that for each arrow f in $\mathcal{B}$, $dom(f)$ and $cod(f)$ are in $\mathbf{Ob}(\mathcal{B})$, along with each composition of arrows, and an identity arrow for each element of $\mathbf{Ob}(\mathcal{B})$.*

A category of interest is the category $\mathbf{Set}$, which has as objects sets and arrows all total functions, with composition of functions as the composition. Clearly this construct has identity arrows and the associative rule applies, so it is indeed a category. The subcategories of interest to us are subcategories of particular types of data sets, denoted $\mathcal{D}$, with objects similar types of data sets and arrows only the identity arrows, and subcategories of particular types of feature sets, denoted $\mathcal{F}$, with objects similar types of feature sets, and arrows only the identity arrows. The objects and arrows of these categories shall correspond to a particular sensor system, so will represent all of the possible data (or feature) sets that can be generated by the sensor-processor system. For example, the data generated by a particular sensor system may be $2x2$ real-valued matrices. In this case, $\mathcal{D} = (\mathbb{R}^{2x2}, \mathbf{id}_{\mathcal{D}}, \mathbf{id}_{\mathcal{D}}, \circ)$ represents the category with only the identities as arrows, and $\circ$ being the usual composition of functions.

Another useful categorical construct is a **functor**.

Definition 3 (Functor) *A **functor** $\mathfrak{F}$ between two categories $\mathcal{A}$ and $\mathcal{B}$ is a pair of mappings $\mathfrak{F} = (\mathfrak{F}_{\mathbf{Ob}}, \mathfrak{F}_{\mathbf{Ar}})$ such that*

$$\mathbf{Ob}(\mathcal{A}) \xrightarrow{\mathfrak{F}_{\mathbf{Ob}}} \mathbf{Ob}(\mathcal{B})$$

$$\mathbf{Ar}(\mathcal{A}) \xrightarrow{\mathfrak{F}_{\mathbf{Ar}}} \mathbf{Ar}(\mathcal{B})$$

while preserving the associative property of the composition map and preserving identity maps.

Thus, given categories $\mathcal{A}, \mathcal{B}$ and functor $\mathfrak{F} : \mathcal{A} \to \mathcal{B}$, if $A \in \mathbf{Ob}(\mathcal{A})$ and $f, g, h, 1_A \in \mathbf{Ar}(\mathcal{A})$ such that $f \circ g = h$ is defined, then there exists $B \in \mathbf{Ob}(\mathcal{B})$ and $f', g', h', 1_B \in \mathbf{Ar}(\mathcal{B})$ such that

(i) $\mathfrak{F}_{\mathbf{Ob}}(A) = B$.

(ii) $\mathfrak{F}_{\mathbf{Ar}}(f) = f',\ \mathfrak{F}_{\mathbf{Ar}}(g) = g'$.

(iii) $h' = \mathfrak{F}_{\mathbf{Ar}}(h) = \mathfrak{F}_{\mathbf{Ar}}(f \circ g) = \mathfrak{F}_{\mathbf{Ar}}(f) \circ \mathfrak{F}_{\mathbf{Ar}}(g) = f' \circ g'$.

(iv) $\mathfrak{F}_{\mathbf{Ar}}(1_A) = 1_{\mathbf{Ob}(A)} = 1_B$.

In general, if a functor between two categories of fusion can be developed or discovered, it could possibly demonstrate an isomorphism between the two. In fact, if a fusion rule in the first category were optimal, and there existed an isomorphic functor to the second, then there might exist an optimal fusion rule in the second fusion category as well (though not necessarily the image of the optimal rule under the functor). Thus, if one fusion category were well understood and was isomorphic to a second, the second would be well understood.

Finally, we need the definition of a natural transformation between functors.

Definition 4 (Natural Transformation) *Given categories $\mathcal{A}$ and $\mathcal{B}$ and functors $\mathfrak{F}$ and $\mathfrak{G}$ with $\mathcal{A} \xrightarrow{\mathfrak{F}} \mathcal{B}$ and $\mathcal{A} \xrightarrow{\mathfrak{G}} \mathcal{B}$, then a **Natural Transformation** is a family of arrows $\nu = \{\nu_{\mathrm{A}} | \mathrm{A} \in \mathcal{A}\}$ such that for each $f \in \mathbf{Ar}(\mathcal{A})$, $\mathrm{A} \xrightarrow{f} \mathrm{A}'$, $\mathrm{A}' \in \mathcal{A}$, the square*

$$\begin{tikzcd}
\mathfrak{F}(\mathrm{A}) \arrow[r, "\nu_A"] \arrow[d, "\mathfrak{F}(f)"'] & \mathfrak{G}(\mathrm{A}) \arrow[d, "\mathfrak{G}(f)"] \\
\mathfrak{F}(\mathrm{A}') \arrow[r, "\nu_{\mathrm{A}'}"] & \mathfrak{G}(\mathrm{A}')
\end{tikzcd}$$

commutes. We then say the arrows $\nu_{\mathrm{A}}, \nu_{\mathrm{A}'}$ are the components of $\nu : \mathfrak{F} \longrightarrow \mathfrak{G}$, and call ν the natural transformation of $\mathfrak{F}$ to $\mathfrak{G}$.

Definition 5 (Functor Category $\mathcal{A}^{\mathcal{B}}$) *Given categories $\mathcal{A}$ and $\mathcal{B}$, the notation $\mathcal{A}^{\mathcal{B}}$ denotes the category of all functors $\mathfrak{F}$, $\mathcal{B} \xrightarrow{\mathfrak{F}} \mathcal{A}$. This category has all such functors as objects and the natural transformations between them as arrows.*

Definition 6 (Product Category) *Let $\{\mathcal{C}_i\}_{i=1}^n$ be a finite collection of categories. Then*

$$\prod_{i=1}^{n} \mathcal{C}_i = \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_n$$

is the corresponding product category.

3 Main Results

In this section we give mathematical definitions of fusion and integration and demonstrate the similarities and differences between the two.

Let $\mathcal{C}$ be a subcategory of the **Set** category throughout this section. For $n \in \mathbb{N}$ we define the cartesian product $\mathcal{C}^n$ to be

$$\mathcal{C}^n = \underbrace{\mathcal{C} \times \mathcal{C} \times \cdots \times \mathcal{C}}_{n \text{ times}}$$

which can be shown to be another category. (See Adamek [2] for details.) In particular, since $\mathcal{C} = (\mathbf{Ob}(\mathcal{C}), \mathbf{Ar}(\mathcal{C}), \mathbf{Id}(\mathcal{C}), \circ)$ then $\mathcal{C}^n = (\mathbf{Ob}(\mathcal{C})^n, \mathbf{Ar}(\mathcal{C})^n, \mathbf{Id}(\mathcal{C})^n, \circ)$. We wish to investigate cartesian products of $\mathcal{C}$, but do not need to be specific about the number of products. Thus, we define the collection of all cartesian products of $\mathcal{C}$ to be

$$\mathbf{Cart}(\mathcal{C}) = \{\mathcal{C}^n : n \in \mathbb{N}\}.$$

Therefore, given category $\mathcal{A} \in \mathbf{Cart}(\mathcal{C})$ there exists an $n \in \mathbb{N}$ such that $\mathcal{A} = \mathcal{C}^n$.

3.1 Fusion

Definition 7 (Fusion Rule) *Let $\{\mathcal{C}_i\}_{i=0}^N$ be subcategories of a category $\mathcal{A}$. Let $\prod_{i=1}^n \mathcal{C}_i$ be a product category. Then a fusion rule is a functor $\mathfrak{R} \in \mathcal{C}_0^{\prod_{i=1}^n \mathcal{C}_i}$.*

Thus, a fusion rule is a pair of mappings $(\mathfrak{R}_{\mathbf{Ob}}, \mathfrak{R}_{\mathbf{Ar}})$.

A fusor is a fusion rule of an event-decision process which performs by means of a functional on its corresponding ROC curve better than any branch of the graph of the original processes before applying a fusion rule.

Definition 8 (Fusor, Fusion) *Let $\mathcal{C}$ be a category. Let S be a collection of fusion rules in $\cup_{n=1}^N \{\mathcal{C}_0^{\prod_{i=1}^n \mathcal{C}_i}\}$. Let φ be a nonnegative real-valued functional defined on S. Assume the optimization problem*

$$\max\{\varphi(\mathfrak{R}) : \mathfrak{R} \in \mathsf{S}\}$$

*has a maximizer $\mathfrak{R}^* \in \mathsf{S}$. We say $\mathfrak{R}^*$ is a **fusor** with respect to φ on S. The process of determining a fusor is called **fusion**.*

If the collection contains only two fusion rules, say $\mathsf{S} = \{\mathfrak{R}_1, \mathfrak{R}_2\}$, and if $\varphi(\mathfrak{R}_1) \leq \varphi(\mathfrak{R}_2)$ then we can define a partial ordering $\preceq$ on $\mathcal{C}^{\prod_{i=1}^{n} \mathcal{C}_i}$, (with respect to φ) to be $\mathfrak{R}_1 \preceq \mathfrak{R}_2$ if and only if $\varphi(\mathfrak{R}_1) \leq \varphi(\mathfrak{R}_2)$.

Definition 9 (Homogenous Fusion) *Let $\mathcal{C}$ be a category and define*

$$\mathsf{HR}\,(\mathcal{C}^n, \mathcal{C}) = \{r : A \to C \mid A = C^n \text{ for some } C \in \mathbf{Ob}(\mathcal{C})\}.$$

*Let φ be a nonnegative-valued functional defined on $\mathsf{HR}\,(\mathcal{C}^n, \mathcal{C})$. The process of find a maximizer $r^* \in \mathsf{HR}\,(\mathcal{C}^n, \mathcal{C})$ is called **homogeneous fusion.***

Definition 10 (Heterogenous Fusion) *Fusion that is not homogeneous is said to be heterogeneous fusion.*

3.2 Integration

Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. Let object $A \in \mathbf{Ob}(\mathcal{A})$ and object $B \in \mathbf{Ob}(\mathcal{B})$. Assume there is a mapping $A \xrightarrow{t} B$ that we can think of as the "truth" mapping. We wish to approximate this mapping with a collection of mappings that compose together. For example, suppose there are mappings $A \xrightarrow{f} C$ and $C \xrightarrow{g} B$ such that the composition $g \circ f$ is defined and approximates t. Let ψ be a real-valued functional defined on B^A , then

$$|\psi(g \circ f) - \psi(t)|$$

quantifies the approximation of $g \circ f$ to t. We define the collection of mappings that are composable $\{A \xrightarrow{f} C, C \xrightarrow{g} B\}$ to be a *system*. Assume the mapping $A \xrightarrow{f} C$ existed but no mapping from C to B existed. We define *integration* to be the process of determining a mapping $C \xrightarrow{g^*} B$ such that the approximation error is as small as possible. That is,

$$\min_{g \in C^B} |\psi(g \circ f) - \psi(t)| = |\psi(g^* \circ f) - \psi(t)|\,.$$

We call g^* an *integrator*.

Now we write definitions using rigorous mathematics. Assume a mapping $t \in B^A$ is given as above, as well as a ψ functional. we need cartesian products of categories that may be different. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ be n categories (possibly distinct). Form the cartesian product $\mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_n$. A functor that maps this product into another category $\mathcal{C}$ (possibly distinct also) is an integration rule.

Definition 11 (Series System) *A **series system** (S-system) for $A \xrightarrow{t} B$ is a finite collection of mappings*

$$\{A \xrightarrow{f_1} C_1, C_1 \xrightarrow{f_2} C_2, \cdots, C_{n-1} \xrightarrow{f_n} C_n, C_n \xrightarrow{f_{n+1}} B\}$$

such that $f_{n+1} \circ f_n \circ \cdots \circ f_1 \in B^A$ for some $n \in \mathbb{N}$, for some $C_1 \in \mathcal{C}_2, C_2 \in \mathcal{C}_2, \ldots, C_1 \in \mathcal{C}_n$ for categories $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$.

Definition 12 (Parallel System) *A **parallel system** (P-system) for $A \xrightarrow{t} B$ is a finite collection of mappings*

$$\{A \xrightarrow{f_1} C_1, A \xrightarrow{f_2} C_2, \cdots, A \xrightarrow{f_n} C_n, C_1 \times C_2 \times \cdots \times C_n \xrightarrow{g} B\}$$

such that $g \circ (f_1, f_2, \ldots, f_n\} \in B^A$ for some $n \in \mathbb{N}$, for some $C_1 \in \mathcal{C}_2, C_2 \in \mathcal{C}_2, \ldots, C_1 \in \mathcal{C}_n$ for categories $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$.

Observe that the graph for a series system is a chain. The graph for a parallel system is "fan out- fan in" graph. See the figures. Note that the categories $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ might not be distinct.

Definition 13 (System of systems) *A **PS-system** for $A \xrightarrow{t} B$ is a finite collection of P-system of S-sytems. A **PP-system** (P^2-system) for $A \xrightarrow{t} B$ is a finite collection of P-system of P-sytems.*

Observe that an S-system of S-systems is another S-system. A S-system of P-systems will be considered later. This process can be continued any number of times to get a P^{n}S-system or P^n-system.

Definition 14 (Integration Rule) *Let $\mathcal{S}_1, \mathcal{S}_2$ be two systems that are not connected. An **integration rule** is a mapping that connectes $\mathcal{S}_1$ and $\mathcal{S}_2$ into a single system S_3.*

Definition 15 (Integrator) *Let T be a collection of integration rules for systems $\mathcal{S}_1, \mathcal{S}_2$. Let ψ be a nonnegative real-valued functional defined on T. Assume the optimization problem*

$$\max\{\psi(\mathfrak{I}) : \mathfrak{I} \in \mathsf{T}\}$$

*has a maximizer $\mathfrak{g}^* \in \mathsf{T}$. We say $\mathfrak{g}^*$ is an **integrator** with respect to ψ on T. The process of determining a integrator is called **integration**.*

4 Examples

Example 1. Consider the simple problem of connecting two S-systems. The integration rule i is $\mathrm{D} \xrightarrow{i} \mathrm{F}$ such

$$\mathrm{E} \xrightarrow{s} \mathrm{D}$$

Fig. 1: System 1.

$$\mathrm{F} \xrightarrow{c} \mathrm{L}$$

Fig. 2: System 2.

that we get the system in Equation 1. Hence, the integration rule is a processor, and the final system is an event-decision system.

$$\mathrm{E} \xrightarrow{s} \mathrm{D} \xrightarrow{i} \mathrm{F} \xrightarrow{c} \mathrm{L} \tag{1}$$

Example 2. Integration rule i is a "data" interfacer $\mathrm{D}_1 \xrightarrow{i} \mathrm{D}_2$ so that given two systems such as

$$\mathrm{E} \xrightarrow{s} \mathrm{D}_1$$

and

$$\mathrm{D}_2 \xrightarrow{p} \mathrm{F} \xrightarrow{c} \mathrm{L}$$

we end up with

$$E \xrightarrow{s} D_1 \xrightarrow{i} D_2 \xrightarrow{p} F \xrightarrow{c} L$$

which is the final event-decision system.

Example 3. In this example, we show that integration is fusion. Given a first system

$$\begin{tikzcd}
 & D_1 \\
E \arrow[ur, "s_1"] \arrow[dr, "s_2"'] & \\
 & D_2
\end{tikzcd}$$

and a second system

$$D_3 \xrightarrow{p} F \xrightarrow{c} L$$

then integration rule i is a fusion rule $D_1 \times D_2 \xrightarrow{i} D_3$ so that

$$\begin{tikzcd}
 & D_1 \arrow[dr, no head] & & & & \\
E \arrow[ur, "s_1"] \arrow[dr, "s_2"'] & & \| \arrow[r, dotted, twoheadrightarrow, "i"] & D_3 \arrow[r, "p"] & F \arrow[r, "c"] & L \\
 & D_2 \arrow[ur, no head] & & & &
\end{tikzcd}$$

is the final event-decision system. Thus, fusion is a special case of integration.

Example 4. Given system 1

$$E \xrightarrow{s} D \xrightarrow{p_1} F_1 \xrightarrow{c_1} L$$

and system 2

$$D \xrightarrow{p_2} F_2 \xrightarrow{c_2} L$$

we see that our integration rule needs label (decision) fusion and is

$$\begin{tikzcd}
 & & F_1 \arrow[r, "c_1"] & L \arrow[dr, no head] & & \\
E \arrow[r, "s"] & D \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & & \| \arrow[r, dotted, twoheadrightarrow, "i"] & L \\
 & & F_2 \arrow[r, "c_2"] & L \arrow[ur, no head] & &
\end{tikzcd}$$

Example 5. Here is a nontrivial integrated system that contains feature fusion and label fusion.

$$\begin{tikzcd}
 & & F_1 \arrow[r, "c_1"] & L_1 \arrow[ddrr, no head] & & & \\
 & D_1 \arrow[ur, "p_1"] \arrow[dr, "p_2"] & & & & & \\
E \arrow[ur, "s_1"] \arrow[dr, "s_2"'] & & F_2 \arrow[dr, no head] & & & \| \arrow[r, dotted, twoheadrightarrow, "f_2"] & L_3 \\
 & D_2 \arrow[r, "p_3"] & F_3 \arrow[r, no head] & \| \arrow[r, dotted, twoheadrightarrow, "f_1"] & L_2 \arrow[ur, no head] & &
\end{tikzcd}$$

Example 6. [Mammography]. Suppose sensor s is an x-ray mammography machine. A woman will have 4 x-rays taken (2 different views on each breast). The woman is an outcome e from a population set E. The x-ray mammogram provides 4 x-ray images, called a case. This is the data d; 4 x-ray images. Let D denote the collection of all possible 4-tuples of images. Therefore, $d = s(e)$ and $s : E \to D$. A radiologist (a person trained to find cancerous regions in a mammogram) looks at the case to determine if any region "appears" to be cancerous. (They look for certain features and characteristics) This region is called a region of interest (ROI) in the medical community. A biopsy is performed to determine if the tissue in the ROI is cancerous or not, thus, the biopsy procedure labels the region of interest. Radiologists desire to able to look at a case and determine for certain if cancer is present. But, radiologists are not perfect, not consistent and not tireless. They may miss a cancer (a type II error called a false negative). A false negative error is very costly for the woman, whereas for a false positive the woman goes through a painful biopsy and mental stress to be relieved that there is no cancer. (Of course, she has to pay for this procedure after all is done, but the cost is less than her life) Some health care related companies are working on devices that aid the radiologist to minimize these errors. Some devices try to emulate the radiologists. Radiologists will look for regions where microcalcifications occur and also regions where breast tissue is dense. Some cancer will form spiculated massy tissue, thus radiologists look for these regions as well.

$$\begin{tikzcd}
 & & F_1 \arrow[r, "c_1"] & L_1 \arrow[dr, no head] & & \\
E \arrow[r, "s"] & D \arrow[ur, "p_1"] \arrow[r, "p_2"] & F_2 \arrow[r, "c_2"] \arrow[dr, "c_3"] & L_2 \arrow[r, no head] & \| \arrow[r, dotted, twoheadrightarrow, "f"] & L_4 \\
 & & & L_3 \arrow[ur, no head] & &
\end{tikzcd}$$

5 Conclusion

Integration can be defined without having to specific the systems' elements. Fusion is the process of mapping several objects to a single object in an optimal fashion. Integration is the process of connecting systems into a larger system. Thses systems may have fusion in them.

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